

Spin tomography

G. M. D'Ariano, L. Maccone,* and M. Panni

*Quantum Optics & Information Group, INFN UdR Pavia,
Dipartimento di Fisica “A. Volta” and I.N.F.M., Via Bassi 6, 27100 Pavia, Italy.*

(Dated: January 29, 2003)

We propose a tomographic reconstruction scheme for spin states. The experimental setup, which is a modification of the Stern–Gerlach scheme, can be easily performed with currently available technology. The method is generalized to multi-particle states, analyzing the spin 1/2 case for indistinguishable particles. Some Monte Carlo numerical simulations are given to illustrate the technique.

PACS numbers: 75.40.G, 03.65.W

I. INTRODUCTION

The main idea of tomography is to reconstruct the density matrix ϱ or, equivalently, the expectation value of any observable of the system from repeated measurements on an ensemble of identical states. In this paper a “spin tomography” for reconstructing spin states is developed in the framework of generalized tomography, starting from group theory [1]. There have been other proposals to infer the spin state [2]. Our method is both easy to carry out experimentally and for the first time allows also a reconstruction of indistinguishable multi-particle spin 1/2 states, which is quite general since it encloses a vast class of experimentally accessible systems.

The best known quantum tomographic procedure is optical homodyne tomography [3], for the reconstruction of the density matrix ϱ_r of the radiation field from the homodyne probability $p(x, \phi)$. It is based on the following formula [4]

$$\varrho_r = \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{+\infty} dx p(x, \phi) K(x - x_\phi), \quad (1)$$

where x_ϕ is the quadrature operator and $K(x)$ is an appropriate kernel function. We will not go into details about this formula, as we only want to stress the analogy with the spin case. In fact, consider the spin density operator ϱ_s , which is defined on a Hilbert space \mathcal{H}_s of dimension $2s + 1$. We will prove the following formula

$$\varrho_s = \sum_{m=-s}^s \int \frac{d\vec{n}}{4\pi} p(\vec{n}, m) K_s(m - \vec{s} \cdot \vec{n}), \quad (2)$$

where the integral is performed over all directions of the vector \vec{n} , $p(\vec{n}, m)$ is the probability of having outcome m measuring the self-adjoint operator $\vec{s} \cdot \vec{n}$ (\vec{s} being the spin operator), and $K_s(x)$ is a kernel function that will be defined later.

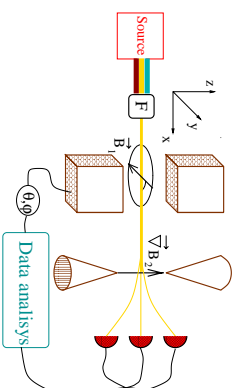


FIG. 1: Experimental apparatus for spin tomography. The Fizeau filter (F in the figure) selects particles with the same velocity from an incoming beam. These are then injected into a magnetic field \vec{B}_1 , which forms an angle φ with the y -axis and has intensity proportional to ϑ . For the tomographic reconstruction the phases ϑ and φ need to be varied during the experiment. The remaining part of the apparatus is a conventional Stern–Gerlach scheme (in the figure we show the case of spin $s = 1$ as an example). A computer finally correlates the experimental results with the parameters ϑ and φ , in order to reconstruct the density matrix, according to formula (6).

It is possible to show that both Eq. (1) and (2) follow from a single operator identity, derived using group theory. In fact, define Tomographic Group G an unimodular group (*i.e.*, left and right invariant measure are coincident) which has a unitary irreducible square-integrable representation $\mathcal{R}(g)$ ($g \in G$) on the Hilbert space \mathcal{H} of the physical system. [A square-integrable representation is such that $\int dg [\langle u | \mathcal{R}(g) | v \rangle]^2 < \infty$, where the integral is extended to all the elements g of the group G , dg is an invariant measure for G , and where the integral is not dependent on the choice of $|u\rangle, |v\rangle \in \mathcal{H}$ (as will be shown later)]. The operator identity we derive (1) and (2) from, and which is valid for any tomographic group G , is the following

$$\varrho = \int dg \text{Tr}[\varrho \mathcal{R}(g)] \mathcal{R}^\dagger(g), \quad (3)$$

valid for any trace class operator ϱ acting on \mathcal{H} . This formula, derived from general considerations in [1, 5] is derived also in App. A using only group theory.

The outline of the paper follows. In Sect. II, the tomography procedure to reconstruct the spin state of sin-

*also at the Massachusetts Institute of Technology, Research Laboratory of Electronics, MIT 36-497, Cambridge, MA 02139, USA.

gle particle systems is introduced and analyzed. The experimental setup is described and some demonstrative numerical simulations of the procedure are studied. In Sect. III the extension to the reconstruction of multi-particle spin states is studied. For distinguishable particles, the general reconstruction procedure is given, while, for indistinguishable particles, the cases of two and three spin 1/2 particles is analyzed in detail. In Sect. IV the orders of magnitude of possible experimental setups are discussed showing the feasibility of the proposed method. In App. A the group tomography is derived by proving the tomographic reconstruction formula (3) in the framework of group theory.

II. SINGLE PARTICLE SPIN TOMOGRAPHY

Starting from the general operator identity (3), we now specify the physical system as a single spin. In this case $\mathcal{H} = \mathbb{C}^{2s+1}$, s being the spin of the particle. For such a system, we can choose the group $SU(2)$ of 2×2 unitary matrices with unit determinant as tomographic group G . In fact, $SU(2)$ can be parametrized through the “rotation parameters” (\vec{n}, ψ) —where $\vec{n} = (\cos \varphi \sin \vartheta, \sin \varphi \sin \vartheta, \cos \vartheta)$, $\vartheta \in [0, \pi]$, $\varphi \in [0, 2\pi]$, and $\psi \in [0, 2\pi]$ — and it induces a unitary irreducible representation on \mathbb{C}^{2s+1} . The operators constituting this representation are given by

$$\mathcal{R}(\vec{n}, \psi) = e^{i\vec{s} \cdot \vec{n} \psi}, \quad (4)$$

where \vec{s} is the particle spin operator. Haar’s invariant measure [6] for $SU(2)$ is, with this parametrization and with the normalization needed for the invariant measure (see App. A),

$$dg(\vec{n}, \psi) = \frac{2s+1}{4\pi^2} \sin^2 \frac{\psi}{2} \sin \vartheta \, d\psi d\vartheta d\varphi. \quad (5)$$

As will be seen in the following, the choice of $SU(2)$ as tomographic group G is not unique. It is easy to obtain the spin tomography (2) starting from Eq. (3), which now rewrites as

$$\begin{aligned} \varrho &= \frac{(2s+1)}{4\pi^2} \int_0^{2\pi} d\psi \sin^2 \frac{\psi}{2} \int_0^\pi d\vartheta \sin \vartheta \\ &\times \int_0^{2\pi} d\varphi \operatorname{Tr} [\varrho e^{i\vec{s} \cdot \vec{n} \psi}] e^{-i\vec{s} \cdot \vec{n} \psi}. \end{aligned} \quad (6)$$

Evaluating the trace over the complete set of vectors $|\vec{n}, m\rangle$ (which are the eigenstates of $\vec{s} \cdot \vec{n}$, relative to the eigenvalue m), we find Eq. (2), by defining

$$K_s(x) \doteq \frac{(2s+1)}{\pi} \int_0^{2\pi} d\psi \sin^2 \frac{\psi}{2} e^{i\psi x}, \quad (7)$$

and by noticing that $\langle \vec{n}, m | \varrho | \vec{n}, m \rangle = p(\vec{n}, m)$.

It should be pointed out that formula (1) for optical homodyne tomography can be proven from Eq. (3) with a very similar argument.

How do we use formula (2)? In order to measure the matrix elements $\varrho_{il} = \langle a_i | \varrho | a_l \rangle$ for all i, l ($\{|a_i\rangle\}$ being a basis for \mathcal{H}_s), we only need to calculate $\langle a_i | K_s(m - \vec{s} \cdot \vec{n}) | a_l \rangle$ and to measure $p(\vec{n}, m)$.

The most convenient choice for the basis $\{|a_i\rangle\}$ is the set $\{|m\rangle\}$ of eigenvectors of s_z ($m = -s, \dots, s$). Thus, the calculation of the matrix elements of the kernel operator, by defining $\lambda_{l,m} \doteq \langle l | \vec{n}, m \rangle$, yields

$$\begin{aligned} &\langle i | K_s(m - \vec{s} \cdot \vec{n}) | l \rangle \\ &= \frac{(2s+1)}{\pi} \int_0^{2\pi} d\psi \sin^2 \frac{\psi}{2} \sum_{m'=-s}^s e^{i\psi(m-m')} \lambda_{i,m'} \lambda_{l,m'}^* \\ &= (2s+1) \left(\lambda_{i,m} \lambda_{l,m}^* - \frac{\lambda_{i,m+1} \lambda_{l,m+1}^* + \lambda_{i,m-1} \lambda_{l,m-1}^*}{2} \right). \end{aligned} \quad (8)$$

Observing that

$$|\vec{n}, m\rangle = e^{-i\vartheta \vec{s} \cdot \vec{n}_\perp} |m\rangle, \quad (9)$$

with $\vec{n}_\perp \doteq (-\sin \varphi, \cos \varphi, 0)$, the evaluation of $\lambda_{l,m}$ is given by

$$\begin{aligned} \lambda_{l,m} &= \langle l | e^{i\vartheta(\sin \varphi s_x - \cos \varphi s_y)} | m \rangle = \langle l | e^{-i\varphi s_z} e^{-i\vartheta s_y} e^{i\varphi s_z} | m \rangle \\ &= e^{i\varphi(m-l)} \sqrt{(s+m)!(s-m)!(s+l)!(s-l)!} \\ &\times \sum_\nu \frac{(-1)^\nu (\cos \frac{\vartheta}{2})^{2s+m-l-2\nu} (-\sin \frac{\vartheta}{2})^{l-m+2\nu}}{(s-l-\nu)!(s+m-\nu)!(\nu+l-m)!\nu!}, \end{aligned} \quad (10)$$

where the sum is performed over the values of ν for which the argument of the factorials is non-negative. In the last equality we used Wigner’s formula [7].

A. Experimental setup and state reconstruction procedure

We now describe the method to measure the state ϱ of an ensemble of non-charged particles, giving the details of the experimental apparatus, depicted in Fig. 1. The beam of particles impinges onto a Fizeau filter, which selects one velocity (in the x direction) for the particles. This is needed in order to ensure that each particle spends the same amount of time t in the subsequent region where a magnetic field \vec{B}_1 is present. The field \vec{B}_1 , which is parallel to the xy plane, is chosen so that $\vec{B}_1 = B_1 \vec{n}_\perp = B_1(-\sin \varphi, \cos \varphi, 0)$. In such way, its effect on the spin state ϱ results in the unitary transformation $U^\dagger \varrho U$, with

$$U = \exp[i\gamma B(\sin \varphi s_x - \cos \varphi s_y)t]. \quad (11)$$

Equation (11) follows from the Hamiltonian $H = -\vec{\mu} \cdot \vec{B}$, with $\vec{\mu} \doteq \gamma \hbar \vec{s}$ ($\vec{\mu}$ being the intrinsic magnetic moment of the particle, and γ its gyromagnetic factor). Successively, the particles cross a gradient of magnetic field \vec{B}_2 , whose effect is to split the beam, giving a measure of s_z for the state $U^\dagger \varrho U$, as in a Stern–Gerlach experiment. In this way we obtain the probability $\langle m | U^\dagger \varrho U | m \rangle$, which is equal to $p(\vec{n}, m)$ by choosing $B_1 = -\vartheta/(\gamma t)$, and by

using Eq. (9). Therefore, by controlling the field \vec{B}_1 , we obtain $p(\vec{n}, m)$ for all \vec{n} . In fact the direction of \vec{B}_1 selects φ , while its intensity B_1 selects ϑ . Now, in order to reconstruct the density matrix ϱ , only data analysis is needed, *i.e.* the insertion of the measured $p(\vec{n}, m)$ into Eq. (2). One may object that an infinite number of measures are required. However, the calculation of the integral in (2) with Monte Carlo techniques guarantees that the reconstructed matrix elements are affected by statistical errors only, which can be made arbitrarily small by increasing the number of measures. In practice, a rather small number of data is required to obtain negligible errors, as we will show by numerically simulating the experiment.

We first simulate the case of a coherent spin state [8], *i.e.*

$$|\alpha\rangle_s \doteq e^{\alpha s + -\alpha^* s -} | -s \rangle, \quad \alpha \in \mathbb{C}, \quad (12)$$

where $s_{\pm} \doteq s_x \pm i s_y$. Notice the similarity with the customary optical coherent state, defined as $|\alpha\rangle \doteq e^{\alpha a^\dagger - \alpha^* a} |0\rangle$, where a is the annihilator operator for the optical mode and $|0\rangle$ is the vacuum state. In Figs. 2 and 3 we show the reconstructed density matrix $\varrho_{coh} = |\alpha\rangle_s \langle \alpha|$ resulting from a Monte Carlo simulated experiment.

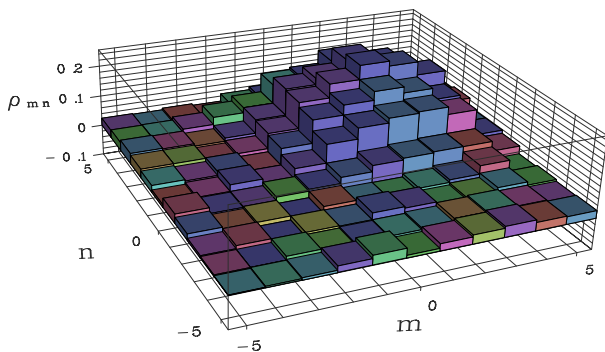


FIG. 2: Simulation of the reconstruction of the density matrix for a coherent spin state ϱ_{coh} . The parameters for the state are $\alpha = 1$ and $s = 5$. The simulation is performed using 3000 spin measurements to generate the density matrix.

As an additional example, in Figs. 4 and 5 we give the simulated reconstruction of a thermal spin state, which is the mixture defined by

$$\varrho_{th} \doteq \frac{e^{-\epsilon s_z}}{\text{Tr}[e^{-\epsilon s_z}]}, \quad \epsilon \in \mathbb{R}. \quad (13)$$

The state ϱ_{th} describes a gas of non interacting spins in thermal equilibrium with a reservoir at a temperature T and in the presence of a magnetic field B_z parallel to the z -axis, *i.e.* $\epsilon = -\gamma \hbar B_z / (K_B T)$, K_B being the Boltzmann constant.

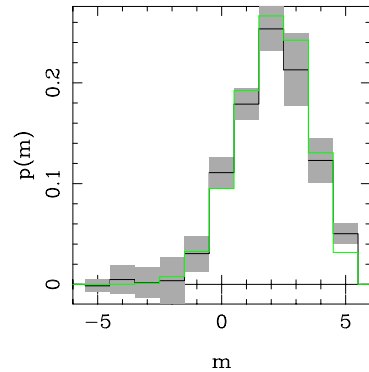


FIG. 3: Diagonal elements of the matrix given in Fig. 2. The statistical error bars are obtained by dividing the measurements into 10 statistical blocks. The solid line indicates the theoretical value.

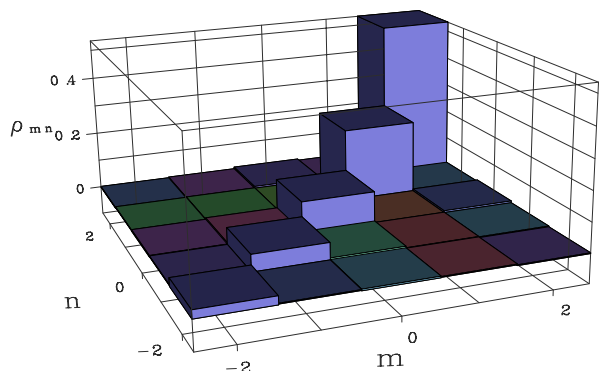


FIG. 4: Density matrix for a thermal spin state ϱ_{th} . Here $\epsilon = .75$ and $s = 2$. A number of 60000 simulated measurements have been used in the reconstruction.

B. Discrete spin tomography

Up to now $SU(2)$ has been used as tomographic group G for the reconstruction of the spin density matrix. This choice for G is not unique. For example, in the case of spin $s = \frac{1}{2}$, it is possible to use also the group defined as $\mathcal{G} \doteq \{i\vec{\sigma}, -i\vec{\sigma}, I, -I\}$, where $\vec{\sigma}$ is the vector of Pauli matrices $\vec{\sigma} \doteq (\sigma_x, \sigma_y, \sigma_z)$. The following irreducible unitary representation on \mathbb{C}^2 exists

$$\begin{aligned} \mathcal{R}(i\sigma_\alpha) &= \mathcal{R}(-i\sigma_\alpha) = \sigma_\alpha, \quad \alpha = x, y, z \\ \mathcal{R}(I) &= \mathcal{R}(-I) = I. \end{aligned} \quad (14)$$

Using this representation, from the tomographic reconstruction formula (3) we obtain

$$\varrho = \sum_{m=-1/2}^{1/2} \sum_{\alpha=x,y,z} p(\vec{n}_\alpha, m) m \sigma_\alpha + \frac{1}{2}. \quad (15)$$

Notice that, by using Eq. (15) it is sufficient to measure the spin in only three directions.

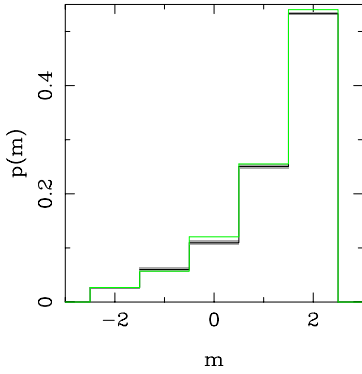


FIG. 5: Main diagonal of the matrix given in Fig. 4. The error bars, which in this case are practically negligible, are obtained by dividing the measurements into 10 statistical blocks. The solid line indicates the theoretical value.

Analogously, for spin $s = 1$ it is possible to find a finite group in alternative to $SU(2)$. In fact, consider the 12 element tetrahedral group composed of the $\pm \frac{2}{3}\pi$ rotations around the versors $\{\vec{n}_1 = \frac{1}{\sqrt{3}}(1, 1, 1), \vec{n}_2 = \frac{1}{\sqrt{3}}(1, -1, -1), \vec{n}_3 = \frac{1}{\sqrt{3}}(-1, 1, -1), \vec{n}_4 = \frac{1}{\sqrt{3}}(-1, -1, 1)\}$, of the π rotations around $\{\vec{n}_5 = (1, 0, 0), \vec{n}_6 = (0, 1, 0), \vec{n}_7 = (0, 0, 1)\}$ and of the identity. It induces a unitary irreducible representation on the space \mathbb{C}^3 , given by the 3×3 rotation matrices. Hence, Eq. (2) now becomes

$$\varrho = \frac{1}{4} \sum_{m=-1}^1 \sum_{j=1}^7 p(\vec{n}_j, m) \mathcal{K}_j(m - \vec{s} \cdot \vec{n}_j) + \frac{1}{4} I, \quad (16)$$

with

$$\mathcal{K}_j(x) = \begin{cases} 2 \cos(\frac{2}{3}\pi x) & j = 1, \dots, 4 \\ e^{-i\pi x} & j = 5, 6, 7 \end{cases}. \quad (17)$$

Notice that this procedure does not make use of a minimal set of measurements, since 14 experimental parameters must be determined in (16), whereas there are only 8 independent real parameters in the 3×3 density matrix. On the contrary, the case of spin $s = \frac{1}{2}$ outlined previously does use the minimal set of measurements for such a system. In Fig. 6 a comparison between the two spin tomography procedures given by Eqs. (6) and (16) is shown through a Monte Carlo simulation. Notice that there is no significant difference in the results, showing that there is no substantial need for a procedure which involves a minimal set of measurements.

For spins $s > 1$ an analogous procedure holds: one needs to find a finite group such that it induces an irreducible unitary representation on $\mathcal{H} = \mathbb{C}^{2s+1}$.

III. MANY PARTICLE SPIN TOMOGRAPHY

The mathematical extension of the method to the case of a system composed of many spins is trivial, yet, it

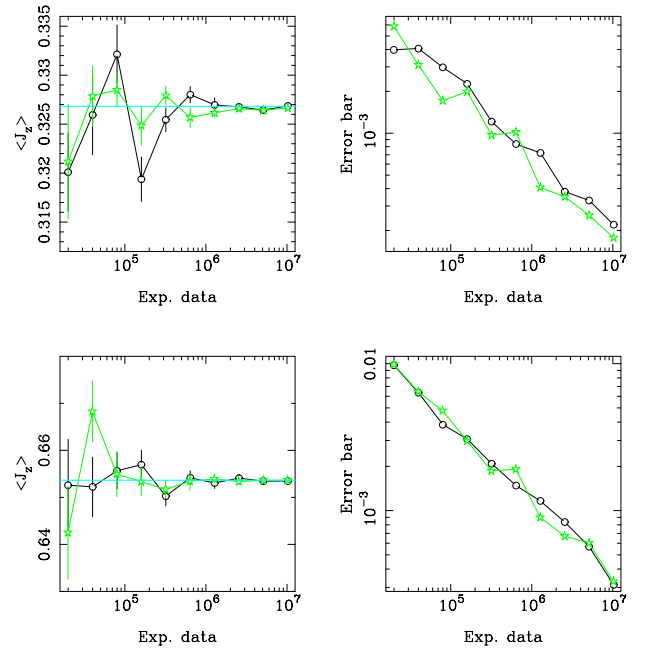


FIG. 6: Monte Carlo comparison between continuous and discrete tomography. Continuous tomography uses $SU(2)$ as tomographic group and is based on Eq. (6), while discrete tomography uses $SU(2)$ finite subgroups and is based on the reconstruction procedures given in Eq. (15) for $s = \frac{1}{2}$ and Eq. (16) for $s = 1$. Left: Convergence of the mean value of $\langle s_z \rangle$ for a coherent $\alpha = 2$ spin state for increasing number of experimental data (the theoretical value is given by the horizontal lines). The circles refer to continuous, the stars to discrete tomography. The upper graph is for spin $s = \frac{1}{2}$, the lower is for $s = 1$. Right: Plots of the statistical error bars of the graphs on the left *vs* experimental data. The error bars are obtained by dividing the experimental data into 20 statistical blocks. Notice that the two tomographic procedures are essentially equivalent: they converge in the same way at the same result.

predicts the necessity of performing measurements on single components and this may not always be possible when the system is composed of indistinguishable particles. For this reason, we need to develop the theory more.

A. Distinguishable spins.

As tomographic group for a system of N spins we can simply use $SU(2)^{\otimes N}$. Up to equivalences, its irreducible representations are given by the direct product of N operators (4) and the invariant measure is the product of N measures (5). As a consequence of the tomography reconstruction formula (3) applied to $SU(2)^{\otimes N}$, we attain

readily the following generalization of Eq. (6)

$$\varrho = \prod_{k=1}^N \frac{(2s_k + 1)}{4\pi^2} \int_0^{2\pi} d\psi_k \sin^2 \frac{\psi_k}{2} \int_0^\pi d\vartheta_k \sin \vartheta_k \times \int_0^{2\pi} d\varphi_k \text{Tr} [\varrho e^{i\vec{s}_k \cdot \vec{n}_k \psi_k}] e^{-i\vec{s}_k \cdot \vec{n}_k \psi_k}, \quad (18)$$

where k is the particle index. The trace term in (18) gives rise to the probability $p(\vec{n}_1, m_1; \dots; \vec{n}_N, m_N)$ of obtaining m_k as result for the measurement of the k th spin \vec{s}_k in the direction \vec{n}_k . This information is accessible only in the case of fully distinguishable spins.

In Fig. 7 a simulated tomographic reconstruction of the value of $\langle S_z \rangle$ (S_z being the total spin component in the z -direction) is given for different multiparticle spin states. Notice how the number of the necessary experimental data increases exponentially with the number of spins, since the statistical error is exponential in the number of particles.

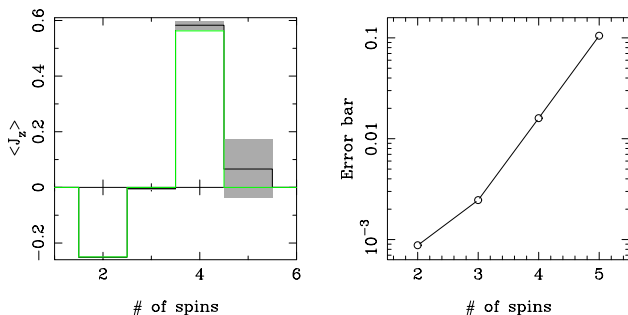


FIG. 7: Left: Plot of $\langle S_z \rangle$ for different number of spins in a completely symmetrical state. A total of 10^6 measurements for each mean value was performed in this simulation. Right: Semilog plot of the error bars *vs* the number of spins. Notice the exponential increase in the statistical errors.

B. Indistinguishable spin 1/2 particles.

Suppose we were given a system of N particles with the same spin. Such particles may be treated as identical by introducing a new dynamical variable, as in the case of the isospin. The spin density matrix (which is the partial trace over the orbital degrees of freedom of the global density matrix) is completely symmetrical, *i.e.*

$$P\varrho P^{-1} = \varrho, \quad (19)$$

for any particle permutation P , because of the complete symmetry of the global density matrix.

It is also possible to see that the spin density matrix is block diagonal in the representation of vectors of definite symmetry, with the subspace corresponding to each block spanned by vectors belonging to the same symmetry. In fact, given $|\phi\rangle$ and $|\psi\rangle$ vectors of different symmetry type

[9], then $\langle\phi|\psi\rangle = 0$. Hence, for any operator ϱ , satisfying (19), one has $\langle\phi|\varrho|\psi\rangle = 0$, as $|\psi\rangle$ belongs to the same symmetry type as $|\psi\rangle$.

Since the square of the total spin S^2 and its z component S_z both commute with all permutation operators P , the common eigenvectors of S^2 and S_z may be taken as a base for each of the diagonal blocks of the spin matrix. Let us now restrict our attention to $s = 1/2$ spin particles. In this case, to each symmetry type there corresponds only one value of S , where $S(S+1)$ is the eigenvalue of S^2 . In fact, given $[\lambda_1\lambda_2]$ the partition of N which defines the class of permutations P that indicate a symmetry type, we find $S = \frac{1}{2}(\lambda_1 - \lambda_2)$ [10].

Let $\mathcal{H}_{S,M}$ be the space of vectors with assigned S and M (M being the eigenvalue of S_z). The spin density matrix restricted to $\mathcal{H}_{S,M}$, which is given by $\varrho_{S,M}$, is again completely symmetrical, hence $[P, \varrho_{S,M}] = 0$. Moreover, $\mathcal{H}_{S,M}$ is associated with an irreducible representation of the permutations group [10]. By using Schur's lemma, we can thus conclude that $\varrho_{S,M} \propto I$, I being the identity in $\mathcal{H}_{S,M}$. In $\mathcal{H}_{S,M}$ there may be vectors of different symmetry type i , yet $\langle i, S, M | \varrho_{S,M} | i, S, M \rangle$ does not depend on the index i , so that the probability for the measurement of S^2 and S_z does not depend on the symmetry type. The same conclusion holds for the measurement of S^2 and $\vec{S} \cdot \vec{m}$ for any vector \vec{m} . Hence, from the arbitrariness of \vec{m} , we conclude that blocks with the same S (and different symmetry type) are coincident.

In conclusion, we have proved that in the $\{S^2\}$ representation ϱ is block diagonal, that each block corresponds to a value of S and that blocks with the same S are equal. Remarkably, applying Eq. (2) to each block, we can reconstruct ϱ measuring only the global quantities S^2 and $\vec{S} \cdot \vec{n}$. Some examples will clarify both the theory and the needed experimental setup.

In the case of two spins 1/2, the spin density matrix will be of the form

$$\varrho = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & 0 \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix} \doteq \sigma \oplus \alpha, \quad (20)$$

where the σ block corresponds to the subspace spanned by the eigenstates of $S = 1$ (which are symmetrical with respect to particles permutations), while the α block to the subspace spanned by the only eigenvector of $S = 0$ (anti-symmetrical with respect to permutations). Applying (2) to each block one finds

$$\varrho = \int \frac{d\vec{n}}{4\pi} \sum_{M=-1}^1 p(S=1, \vec{S} \cdot \vec{n} = M) K_{s=1}(M - \vec{S} \cdot \vec{n}) \oplus p(S=0). \quad (21)$$

According to (21), in order to measure ϱ , we only need the probability distributions $p(S, \vec{S} \cdot \vec{n})$, corresponding to the operators S^2 and $\vec{S} \cdot \vec{n}$, for all \vec{n} , which can be suitably recovered using the apparatus depicted in Fig. 8, which will be analyzed later.

Similarly, the spin density matrix of three spins $1/2$ is

$$\varrho = \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} & 0 & 0 & 0 & 0 \\ \xi_{21} & \xi_{22} & \xi_{23} & \xi_{24} & 0 & 0 & 0 & 0 \\ \xi_{31} & \xi_{32} & \xi_{33} & \xi_{34} & 0 & 0 & 0 & 0 \\ \xi_{41} & \xi_{42} & \xi_{43} & \xi_{44} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi_{11}^1 & \pi_{12}^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi_{21}^1 & \pi_{22}^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pi_{11}^2 & \pi_{12}^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pi_{21}^2 & \pi_{22}^2 \end{pmatrix}. \quad (22)$$

The ξ block corresponds to $S = 3/2$, whereas the π blocks both correspond to $S = 1/2$, and are distinguished by their different symmetry properties. The argument presented previously proves that $\pi_{ij}^1 = \pi_{ji}^1$, for all i, j , thus we can write $\varrho = \xi \oplus \pi \oplus \pi$, with $\pi \doteq \pi^1 = \pi^2$. Again, applying (2) to each block leads to

$$\xi = \int \frac{d\vec{n}}{4\pi} \sum_{M=-\frac{3}{2}}^{\frac{3}{2}} p(S = \frac{3}{2}, \vec{S} \cdot \vec{n} = M) \times K_{s=\frac{3}{2}}(M - \vec{S} \cdot \vec{n}), \quad (23)$$

$$\pi = \int \frac{d\vec{n}}{4\pi} \sum_{M=-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} p(S = \frac{1}{2}, \vec{S} \cdot \vec{n} = M) \times K_{s=\frac{1}{2}}(M - \vec{S} \cdot \vec{n}), \quad (24)$$

and the problem of determining ϱ is again reconducted to the simultaneous measurement of S^2 and $\vec{S} \cdot \vec{n}$.

Both in the cases presented and in the general n spins case, the required experimental data are the distributions $p(S, \vec{S} \cdot \vec{n})$. The apparatus to produce such data are basically equivalent in the two cases, as evident in Figs. 8 and 9, hence we shall limit the analysis to the two spins case. Here, the Fizeau filter and the magnetic field $\vec{B}_1 = B_1 \vec{n}_\perp = B_1(-\sin \varphi, \cos \varphi, 0)$ have the same purpose as in single particle tomography (Fig. 1).

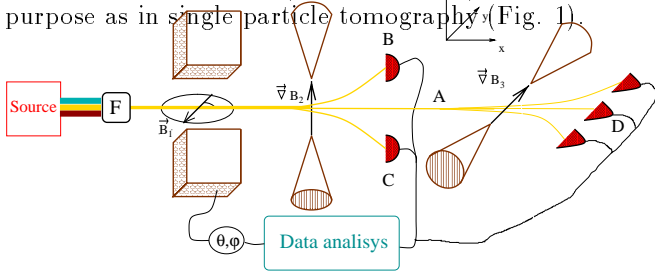


FIG. 8: Experimental apparatus for the tomography of systems composed of two spins $\frac{1}{2}$.

Consider a beam of n non-interacting systems composed of two particles with spin $1/2$. As the analysis can be immediately extended to a mixed case, for simplicity let us consider each system in the pure state

$$|\Psi_o\rangle = \gamma^s |0, 0\rangle + \gamma_{-1}^a |1, -1\rangle + \gamma_0^a |1, 0\rangle + \gamma_1^a |1, 1\rangle, \quad (25)$$

with $|a, b\rangle$ standing for $|S = a, M = b\rangle$. The beam is split into three parts by the gradient \vec{B}_2 , and the systems arrive in detector B with a probability $p(S = 1, M = 1) = |\gamma_1^a|^2$ and in detector C with a probability $p(S = 1, M = -1) = |\gamma_{-1}^a|^2$. The remaining particles reach position A with a probability

$$p_A = |\gamma^s|^2 + |\gamma_0^a|^2 \quad (26)$$

and are left in the state

$$|\Psi_A\rangle = \frac{1}{p_A} \xi (\gamma^s |0, 0\rangle + \gamma_0^a |1, 0\rangle). \quad (27)$$

As the subsequent gradient is directed along the y axis, Eq. (27) is conveniently written using the eigenstates of S_y , i.e. $|S, M\rangle_y$:

$$|\Psi_A\rangle = \frac{1}{p_A} [\gamma^s |0, 0\rangle_y + \gamma_0^a \alpha_{-1} |1, -1\rangle_y + \gamma_0^a \alpha_0 |1, 0\rangle_y + \gamma_0^a \alpha_1 |1, 1\rangle_y], \quad (28)$$

where $\alpha_i \doteq {}_y\langle 1, i | 1, 0 \rangle$ ($i = -1, 0, 1$). Hence, the probability for a system to arrive at detector D is

$$p_S = \frac{1}{p_A} [|\gamma^s|^2 + |\gamma_0^a|^2 |\alpha_0|^2]. \quad (29)$$

By measuring p_A and p_S , the quantities $|\gamma^s|^2$ and $|\gamma_0^a|^2$ are obtained by inverting equations (26) and (29). The coefficients $|\gamma_0^a|^2$, $|\gamma^s|^2$, $|\gamma_1^a|^2$ and $|\gamma_{-1}^a|^2$ are the four probabilities $p(S, M)$ we need for the reconstruction given by Eq. (21).

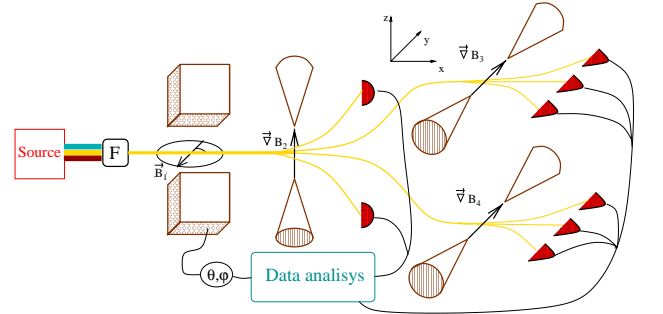


FIG. 9: Experimental apparatus for the tomographic reconstruction of the spin states of systems composed of three spin $\frac{1}{2}$ states.

A similar argument shows that the equipment of Fig. 9 supplies $p(S, M)$, for all S, M , for a system constituted of three spins $1/2$.

IV. FEASIBILITY

The orders of magnitude of the experimental parameters are such that the experiment is feasible with currently available technology. Only as an example, consider

the following cases of spin measurements of electrons or nucleons. For the magnet which is responsible for the field \vec{B}_1 with length of the order of 1 cm, we can measure the state of a beam of electrons with speed $\sim 10^9$ cm/sec, by using a magnetic field $B_1 = \vartheta/\gamma t$ varying between 0 and ~ 30 Gauss. On the other hand, in the nucleon case, choosing a speed of $\sim 10^7$ cm/sec, we need B_1 ranging between 0 to $\sim 10^2 \div 10^3$ Gauss. Obviously, the parameters B_1 and t can be adjusted over a wide range, according to the experimental situation.

V. CONCLUSIONS

We have presented a tomographic experimental procedure for the measurement of the spin density matrix. The experimental scheme is a consequence of formula (2), which was proved using group theory. Through some Monte Carlo simulations, we have shown that the reconstruction can be achieved with high precision using a limited number of measurements. The extension of the procedure to the reconstruction of states of multiple spin systems has been given, both for distinguishable spins and for indistinguishable spin 1/2 particles. Finally, we have shown that the orders of magnitude for the experimental setup are such that it can be implemented with currently available technology.

Acknowledgments

This work has been sponsored by the INFN through the project PRA-2002-CLON and by the EEC through the project IST-2000-29681 (ATESIT).

APPENDIX A: GROUP DERIVATION OF QUANTUM TOMOGRAPHY

The proof of Eq. (3) is obtained from the following lemma.

Let A be an arbitrary trace-class operator on the Hilbert space \mathcal{H} of the system and \mathcal{R} an irreducible unitary square integrable representation on \mathcal{H} of the tomographic group G . Then

$$\text{Tr} A = \int dg \mathcal{R}(g) A \mathcal{R}^\dagger(g), \quad (\text{A1})$$

where dg is an invariant measure for the group G , normalized as $\int dg |\langle u | \mathcal{R}(g) | v \rangle|^2 = 1$ which is independent on the choice of the vectors $|u\rangle, |v\rangle \in \mathcal{H}$.

Proof: By using the unitarity of \mathcal{R} and the properties of group representations, we can write for any $h \in G$

$$\begin{aligned} & \int dg \mathcal{R}(g) |u\rangle \langle v| \mathcal{R}^\dagger(g) \mathcal{R}(h) \\ &= \int dg' \mathcal{R}(hg') |u\rangle \langle v| \mathcal{R}^\dagger(g') \end{aligned} \quad (\text{A2})$$

$$= \mathcal{R}(h) \int dg' \mathcal{R}(g') |u\rangle \langle v| \mathcal{R}^\dagger(g'),$$

which, through Schur's lemma, guarantees that

$$\int dg \mathcal{R}(g) |u\rangle \langle v| \mathcal{R}^\dagger(g) = \tau_{u,v} I_{\mathcal{H}}, \quad (\text{A3})$$

$I_{\mathcal{H}}$ being the identity in \mathcal{H} . Consider the quantity

$$\int dg |\langle u | \mathcal{R}(g) | v \rangle|^2 = \tau_{v,v}, \quad (\text{A4})$$

where $|u\rangle$ and $|v\rangle$ are arbitrary normalized vectors in \mathcal{H} . From Eq. (A3) it is trivial to see that $\tau_{v,v}$ is independent on $|u\rangle$. One can check that it is also independent on $|v\rangle$ by noticing that, given an arbitrary vector $|a\rangle$

$$\begin{aligned} \tau_{v,v} &= \int dg \mathcal{R}(g) |v\rangle \langle v| \mathcal{R}^\dagger(g) = \int dh \mathcal{R}(h^{-1}) |v\rangle \langle v| \mathcal{R}^\dagger(h^{-1}) \\ &= \int dh \mathcal{R}(h) |a\rangle \langle a| \mathcal{R}^\dagger(h) = \tau_{a,a}, \end{aligned} \quad (\text{A5})$$

where the group unimodularity has been used in $dg = dh$ with $h \doteq g^{-1}$. Notice that the hypothesis of square-integrability of the representation guarantees the convergence of the integral in (A4). Thus, the natural choice for the normalization of the group's measure is to take $\tau_{v,v} = 1$. The constant $\tau_{u,v}$ can be expressed in terms of $\tau_{v,v}$ by noticing that upon taking $h \doteq g^{-1}$ one has

$$\begin{aligned} 1 &= \tau_{v,v} = \int dg \langle b | \mathcal{R}(g) | a \rangle \langle a | \mathcal{R}^\dagger(g) | b \rangle = \\ &= \int dh \frac{\langle a | \mathcal{R}(h) | u \rangle \langle v | \mathcal{R}^\dagger(h) | a \rangle}{\langle v | u \rangle} = \frac{\tau_{u,v}}{\langle v | u \rangle}. \end{aligned} \quad (\text{A6})$$

The lemma's thesis is now easily found by using the Schmidt decomposition of A as $A = \sum_i \alpha_i |u_i\rangle \langle v_i|$:

$$\begin{aligned} \int dg \mathcal{R}(g) A \mathcal{R}^\dagger(g) &= \sum_i \alpha_i \int dg \mathcal{R}(g) |u_i\rangle \langle v_i| \mathcal{R}^\dagger(g) = \\ &= \sum_i \alpha_i |v_i\rangle \langle u_i| = \text{Tr} A. \end{aligned} \quad (\text{A7})$$

Group Tomography Theorem. *Let A be an arbitrary trace-class operator on the Hilbert space \mathcal{H} of the system and \mathcal{R} an irreducible unitary square integrable representation on \mathcal{H} of the tomographic group G . Then*

$$A = \int dg \text{Tr}[A \mathcal{R}(g)] \mathcal{R}^\dagger(g). \quad (\text{A8})$$

Proof: Let O be an invertible trace-class operator, it follows that $\mathcal{R}(g)O$ is trace-class for any $g \in G$. Hence it is possible to obtain, by applying (A1) twice

$$\int dg \text{Tr}[A \mathcal{R}(g)] O \mathcal{R}^\dagger(g) = \int dg' \text{Tr}[\mathcal{R}^\dagger(g') O] \mathcal{R}(g') \quad (\text{A9})$$

Take a basis $\{|k\rangle\}$ in \mathcal{H} , one can obtain, using again lemma (A1),

$$\begin{aligned} \int dg \operatorname{Tr}[\mathcal{R}^\dagger(g)O]\langle i|\mathcal{R}(g)A|j\rangle &= \\ \int dg \sum_k \langle k|\mathcal{R}^\dagger(g)O|k\rangle\langle i|\mathcal{R}(g)A|j\rangle &= \\ \sum_k \langle k|O\operatorname{Tr}[|k\rangle\langle i|]A|j\rangle &= \langle i|OA|j\rangle. \end{aligned} \quad (\text{A10})$$

From Eqs. (A9) and (A10) it follows immediately that $\int dg \operatorname{Tr}[A\mathcal{R}(g)]O\mathcal{R}^\dagger(g) = OA$, which yields the thesis (A8) by multiplying to the left both members by O^{-1} .

It is trivial to extend theorem (A8) to the case of projective representations, *i.e.* group representations for

which, given $g_1, g_2, g_3 \in G$ such that $g_1 \cdot g_2 = g_3$, one has

$$\mathcal{R}(g_1)\mathcal{R}(g_2) = e^{i\zeta(g_1, g_2)}\mathcal{R}(g_3), \quad (\text{A11})$$

$\zeta \in \mathbb{R}$ being a phase factor depending on g_1 and g_2 . Notice, moreover, that the theorem here presented is valid also for discrete and finite groups, with the sum on group elements replacing the integral. From result (A8), with an appropriate choice for the tomographic group and the irreducible representation, it is possible to prove the formula for spin tomography (2) –derived in the following section– and for optical homodyne tomography (1). Notice that the unimodularity hypothesis given in the definition of tomographic group G can be relaxed without losing most of the results we give in this paper.

-
- [1] G. M. D'Ariano, Phys. Lett. A **268** 151 (2000); G. M. D'Ariano, L. Maccone and M. G. A. Paris, Physics Letters A **276** 25 (2000); G. M. D'Ariano, L. Maccone, M. G. A. Paris, Journ. of Phys. A **34**, 93 (2001).
 - [2] J.-P. Amiet, S. Weigert, J. Phys. A **32**, L269 (1999); Weigert, Eprint quant-ph/9904095; R. G. Newton and B.-L. Young, Ann. Phys. (N.Y.) **49**, 393 (1968); U. Leonhardt, Phys. Rev. Lett. **74**, 4101 (1995); U. Leonhardt, Phys. Rev. A **53**, 2998 (1996) and references therein; W. K. Wootters, Found. Phys. **16** 391 (1986); T. Opatrný, D.-G. Welsch, and V. Bužek, Phys. Rev. A **53**, 3822 (1996); W. Band and J. L. Park, Am. J. Phys. **47**, 188 (1979); W. Band and J. L. Park, Found. Phys. **1**, 339 (1971); C. Brif and A. Mann, Phys. Rev. A **59**, 971 (1999).
 - [3] D. T. Smithey, M. Beck, M. G. Raymer and A. Faridani, Phys. Rev. Lett. **70**, 1244 (1993); G. M. D'Ariano, U. Leonhardt, M. Paul, Phys. Rev. A **52** R1801 (1995).
 - [4] G.M. D'Ariano, U. Leonhardt, M. Paul, Phys. Rev. A **52** R1801 (1995); G.M. D'Ariano, *Measuring quantum states*, in *Quantum Optics and the Spectroscopy of Solids*, ed. by T. Hakioglu and A.S. Shumovsky, Kluwer Academic Publishers (1997), p. 175.
 - [5] G. M. D'Ariano, L. Maccone and M. G. A. Paris, Physics Letters A **276** 25 (2000); G. M. D'Ariano, L. Maccone, M. G. A. Paris, Journ. of Phys. A **34**, 93 (2001).
 - [6] Haar's measure $d\mu(g)$ must be such that $\int d\mu(g) = 1$, and $\forall h \in \mathcal{G}$, $d\mu(g) = d\mu(h \cdot g) = d\mu(g \cdot h)$.
 - [7] E. Wigner, *Group theory and its application to the quantum mechanics of atomic spectra*, Academic press (1959), formula (15.27), pg. 167.
 - [8] A. M. Perelomov, Commun. Math. Phys. **26**, 222 (1972); M. Rasetti, Int. J. Theor. Phys. **13**, 425 (1973).
 - [9] M. Hamermesh, *Group theory*, Addison-Welley (1962) chap. 7.
 - [10] A. Messiah, *Quantum Mechanics*, North Holland (1961), App. D.